Introduction to Quantum Computation – Assignments
The University of Tokyo – Summer 2015
Instructor: François Le Gall

Choose at least **two among the three problems** below and solve all the questions in the problems you choose. Submit your report to the instructor **before August 19th (17:00)** by putting a copy in the instructor's mailbox located on the 1st floor of the Faculty of Science Building Number 7. Do not forget to write your name and your student number on the report.

Problem 1

In Lecture 5 we studied quantum teleportation and saw how Alice can teleport a qubit to Bob using two classical bits of communication when they initially share the quantum state $|\Psi_1\rangle_{\mathsf{AB}} = \frac{1}{\sqrt{2}}(|0\rangle_{\mathsf{A}}|0\rangle_{\mathsf{B}} + |1\rangle_{\mathsf{A}}|1\rangle_{\mathsf{B}})$, where the first qubit is owned by Alice and the second qubit is owned by Bob. Answer the following questions.

(1) Show that quantum teleportation can be also done if, instead of $|\Psi_1\rangle_{\mathsf{AB}}$, Alice and Bob initially share any of the three following states:

$$|\Psi_2\rangle_{\mathsf{AB}} = \frac{1}{\sqrt{2}}(|0\rangle_{\mathsf{A}}|0\rangle_{\mathsf{B}} - |1\rangle_{\mathsf{A}}|1\rangle_{\mathsf{B}}),$$

$$|\Psi_3\rangle_{\mathsf{AB}} = \frac{1}{\sqrt{2}}(|0\rangle_{\mathsf{A}}|1\rangle_{\mathsf{B}} + |1\rangle_{\mathsf{A}}|0\rangle_{\mathsf{B}}),$$

$$|\Psi_4\rangle_{\mathsf{AB}} = \frac{1}{\sqrt{2}}(|0\rangle_{\mathsf{A}}|1\rangle_{\mathsf{B}} - |1\rangle_{\mathsf{A}}|0\rangle_{\mathsf{B}}).$$

(2) More generally, show that quantum teleportation can be done if Alice and Bob initially share any state of the form

$$\alpha|0\rangle_{\mathsf{A}}|\varphi\rangle_{\mathsf{B}} + \beta|1\rangle_{\mathsf{A}}|\varphi'\rangle_{\mathsf{B}}$$

for any complex numbers α, β such that $|\alpha|^2 = |\beta|^2 = \frac{1}{2}$ and any orthogonal one-qubit states $|\varphi\rangle$ and $|\varphi'\rangle$.

(3) Suppose that Alice now wants to teleport two qubits to Bob. Show how this can be done using four bits of classical communication when Alice and Bob initially share the quantum state

$$|\Phi\rangle_{\mathsf{AB}} = \frac{1}{2}\left(|00\rangle_{\mathsf{A}}|00\rangle_{\mathsf{B}} + |01\rangle_{\mathsf{A}}|01\rangle_{\mathsf{B}} + |10\rangle_{\mathsf{A}}|10\rangle_{\mathsf{B}} + |11\rangle_{\mathsf{A}}|11\rangle_{\mathsf{B}}\right),$$

where Alice owns the first two quits, and Bob owns the last two qubits. Give other examples of four-qubit states that can be used instead of $|\Phi\rangle_{\mathsf{AB}}$. Try to find a characterization of all the four-qubit states that can be used for teleporting two qubits.

Problem 2
Answer the following two questions.

(1) For any positive integer m, the m-th harmonic number H_m is defined as
$$H_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m-1} + \frac{1}{m}.$$
It is known (and not hard to prove) that $H_m = O(\log m)$. Show that, for any positive integers m and n such that $m \leq 2^n$, the inequality
$$\sqrt{\frac{2^n}{1}} + \sqrt{\frac{2^n}{2}} + \cdots + \sqrt{\frac{2^n}{m-1}} + \sqrt{\frac{2^n}{m}} \leq \sqrt{m 2^n H_m}$$
holds.

(2) Let $f\colon \{0,1\}^n \to \{0,1\}$ be a Boolean function. Let us suppose that, as for Grover's algorithm and most of the problems discussed in this course, the function f is given as a quantum gate O_f acting on $n+1$ qubits in the following way: for any $x \in \{0,1\}^n$ and any $a \in \{0,1\}$, the gate O_f maps the quantum state $|x\rangle|a\rangle$ to the state $|x\rangle|a \oplus f(x)\rangle$, where $\oplus$ denotes the bit parity. Define
$$m = |\{x \in \{0,1\}^n \mid f(x) = 1\}|$$
(i.e., m is the number of non-zero values of f). We suppose that $m > 0$ and that m is known. Describe a quantum algorithm that finds, with high probability, *all* the elements $x \in \{0,1\}^n$ such that $f(x) = 1$ and analyze its complexity using the inequality proved in Question (1). Make a comparison with classical strategies for the same task.

Problem 3
Write a short article (say, about two page long) on a quantum algorithm of your choice other than those explained in details during the course. You can find a useful list of references at the Quantum Algorithms Zoo available at the following URL:

http://math.nist.gov/quantum/zoo/.

Your article does not need to give a complete treatment of the algorithm. It should instead focus on presenting the main ideas and the main techniques.